

# HERMITE-HADAMARD-TYPE INEQUALITIES FOR NEW DIFFERENT KINDS OF CONVEX DOMINATED FUNCTIONS

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**ABSTRACT.** In this paper, we establish several new convex dominated functions and then we obtain new Hadamard type inequalities.

## 1. INTRODUCTION

The inequality

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}$$

which holds for all convex functions  $f : [a, b] \rightarrow \mathbb{R}$ , is known in the literature as Hermite-Hadamard's inequality.

In [7], Toader defined  $m$ -convexity as the following:

**Definition 1.** The function  $f : [0, b] \rightarrow \mathbb{R}$ ,  $b > 0$ , is said to be  $m$ -convex where  $m \in [0, 1]$ , if we have

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

for all  $x, y \in [0, b]$  and  $t \in [0, 1]$ . We say that  $f$  is  $m$ -concave if  $(-f)$  is  $m$ -convex.

In [3], Dragomir proved the following theorem.

Let  $f : [0, \infty) \rightarrow \mathbb{R}$  be an  $m$ -convex function with  $m \in (0, 1]$  and  $0 \leq a < b$ . If  $f \in L_1[a, b]$ , then the following inequalities hold:

$$(1.2) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b \frac{f(x) + mf\left(\frac{x}{m}\right)}{2} dx \\ \leq \frac{1}{2} \left[ \frac{f(a) + mf\left(\frac{a}{m}\right)}{2} + m \frac{f\left(\frac{b}{m}\right) + mf\left(\frac{b}{m^2}\right)}{2} \right].$$

In [4] and [5], the authors connect together some disparate threads through a Hermite-Hadamard motif. The first of these threads is the unifying concept of a  $g$ -convex dominated function. Similarly, in [8], Kavurmacı et al. introduced the following class of functions and then proved a theorem for this class of functions related to (1.2).

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*Date:* February 2, 2012.

*2000 Mathematics Subject Classification.* Primary 26D15, Secondary 26D10, 05C38.

*Key words and phrases.*  $m$ -convex dominated function, Hermite-Hadamard's inequality,  $(\alpha, m)$ -convex function,  $r$ -convex function.

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**Definition 2.** Let  $g : [0, b] \rightarrow \mathbb{R}$  be a given  $m$ -convex function on the interval  $[0, b]$ . The real function  $f : [0, b] \rightarrow \mathbb{R}$  is called  $(g, m)$ -convex dominated on  $[0, b]$  if the following condition is satisfied

$$\begin{aligned} & |\lambda f(x) + m(1 - \lambda)f(y) - f(\lambda x + m(1 - \lambda)y)| \\ & \leq \lambda g(x) + m(1 - \lambda)g(y) - g(\lambda x + m(1 - \lambda)y) \end{aligned}$$

for all  $x, y \in [0, b]$ ,  $\lambda \in [0, 1]$  and  $m \in [0, 1]$ .

**Theorem 1.** Let  $g : [0, \infty) \rightarrow \mathbb{R}$  be an  $m$ -convex function with  $m \in (0, 1]$ .  $f : [0, \infty) \rightarrow \mathbb{R}$  is  $(g, m)$ -convex dominated mapping and  $0 \leq a < b$ . If  $f \in L_1[a, b]$ , then one has the inequalities:

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b \frac{f(x) + mf\left(\frac{x}{m}\right)}{2} dx - f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{1}{b-a} \int_a^b \frac{g(x) + mg\left(\frac{x}{m}\right)}{2} dx - g\left(\frac{a+b}{2}\right) \end{aligned}$$

and

$$\begin{aligned} & \left| \frac{1}{2} \left[ \frac{f(a) + mf\left(\frac{a}{m}\right)}{2} + m \frac{f\left(\frac{b}{m}\right) + mf\left(\frac{b}{m^2}\right)}{2} \right] - \frac{1}{b-a} \int_a^b \frac{f(x) + mf\left(\frac{x}{m}\right)}{2} dx \right| \\ & \leq \frac{1}{2} \left[ \frac{g(a) + mg\left(\frac{a}{m}\right)}{2} + m \frac{g\left(\frac{b}{m}\right) + mg\left(\frac{b}{m^2}\right)}{2} \right] - \frac{1}{b-a} \int_a^b \frac{g(x) + mg\left(\frac{x}{m}\right)}{2} dx. \end{aligned}$$

In [9], definition of  $(\alpha, m)$ -convexity was introduced by Miheşan as the following.

**Definition 3.** The function  $f : [0, b] \rightarrow \mathbb{R}$ ,  $b > 0$ , is said to be  $(\alpha, m)$ -convex, where  $(\alpha, m) \in [0, 1]^2$ , if we have

$$f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha)f(y)$$

for all  $x, y \in [0, b]$  and  $t \in [0, 1]$ .

Denote by  $K_m^\alpha(b)$  the class of all  $(\alpha, m)$ -convex functions on  $[0, b]$  for which  $f(0) \leq 0$ . If we take  $(\alpha, m) = \{(0, 0), (\alpha, 0), (1, 0), (1, m), (1, 1), (\alpha, 1)\}$ , it can be easily seen that  $(\alpha, m)$ -convexity reduces to increasing:  $\alpha$ -starshaped, starshaped,  $m$ -convex, convex and  $\alpha$ -convex, respectively.

In [14], Set et al. proved the following Hadamard type inequalities for  $(\alpha, m)$ -convex functions.

**Theorem 2.** Let  $f : [0, \infty) \rightarrow \mathbb{R}$  be an  $(\alpha, m)$ -convex function with  $(\alpha, m) \in (0, 1]^2$ . If  $0 \leq a < b < \infty$  and  $f \in L_1[a, b] \cap L_1\left[\frac{a}{m}, \frac{b}{m}\right]$ , then the following inequality holds:

$$(1.3) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b \frac{f(x) + m(2^\alpha - 1)f\left(\frac{x}{m}\right)}{2^\alpha} dx.$$

**Theorem 3.** Let  $f : [0, \infty) \rightarrow \mathbb{R}$  be an  $(\alpha, m)$ -convex function with  $(\alpha, m) \in (0, 1]^2$ . If  $0 \leq a < b < \infty$  and  $f \in L_1[a, b]$ , then the following inequality holds:

$$(1.4) \quad \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{1}{2} \left[ \frac{f(a) + f(b) + m\alpha f\left(\frac{a}{m}\right) + m\alpha f\left(\frac{b}{m}\right)}{\alpha + 1} \right].$$

For the recent results based on the above definition see the papers [1], [2], [10], [11] and [13].

In [12], the power mean  $M_r(x, y; \lambda)$  of order  $r$  of positive numbers  $x, y$  is defined by

$$M_r(x, y; \lambda) = \begin{cases} (\lambda x^r + (1-\lambda)y^r)^{\frac{1}{r}}, & r \neq 0 \\ x^\lambda y^{1-\lambda}, & r = 0. \end{cases}$$

A positive function  $f$  is  $r$ -convex on  $[a, b]$  if for all  $x, y \in [a, b]$  and  $\lambda \in [0, 1]$

$$(1.5) \quad f(\lambda x + (1-\lambda)y) \leq M_r(f(x), f(y); \lambda).$$

The generalized logarithmic mean of order  $r$  of positive numbers  $x, y$  is defined by

$$(1.6) \quad L_r(x, y) = \begin{cases} \frac{r}{r+1} \frac{x^{r+1} - y^{r+1}}{x^r - y^r}, & r \neq 0, 1, x \neq y \\ \frac{x-y}{\ln x - \ln y}, & r = 0, x \neq y \\ xy \frac{\ln x - \ln y}{x-y}, & r = -1, x \neq y \\ x, & x = y \end{cases}$$

In [6], the following theorem was proved by Gill et al. for  $r$ -convex functions.

**Theorem 4.** Suppose  $f$  is a positive  $r$ -convex function on  $[a, b]$ . Then

$$(1.7) \quad \frac{1}{b-a} \int_a^b f(x) dx \leq L_r(f(a), f(b)).$$

If  $f$  is a positive  $r$ -concave function, then the inequality is reversed.

In the following sections our main results are given: We establish several new convex dominated functions and then we obtain new Hadamard type inequalities.

## 2. $(g - (\alpha, m))$ -CONVEX DOMINATED FUNCTIONS

**Definition 4.** Let  $g : [0, b] \rightarrow \mathbb{R}$ ,  $b > 0$  be a given  $(\alpha, m)$ -convex function on the interval  $[0, b]$ . The real function  $f : [0, b] \rightarrow \mathbb{R}$  is called  $(g - (\alpha, m))$ -convex dominated on  $[0, b]$  if the following condition is satisfied

$$(2.1) \quad \begin{aligned} & |\lambda^\alpha f(x) + m(1-\lambda^\alpha)f(y) - f(\lambda x + m(1-\lambda)y)| \\ & \leq \lambda^\alpha g(x) + m(1-\lambda^\alpha)g(y) - g(\lambda x + m(1-\lambda)y) \end{aligned}$$

for all  $x, y \in [0, b]$ ,  $\lambda \in [0, 1]$  and  $(\alpha, m) \in [0, 1]^2$ .

The next simple characterisation of  $(\alpha, m)$ -convex dominated functions holds.

**Lemma 1.** Let  $g : [0, b] \rightarrow \mathbb{R}$  be an  $(\alpha, m)$ -convex function on the interval  $[0, b]$  and the function  $f : [0, b] \rightarrow \mathbb{R}$ . The following statements are equivalent:

- (1)  $f$  is  $(g - (\alpha, m))$ -convex dominated on  $[0, b]$ .
- (2) The mappings  $g - f$  and  $g + f$  are  $(\alpha, m)$ -convex functions on  $[0, b]$ .

(3) There exist two  $(\alpha, m)$ -convex mappings  $h, k$  defined on  $[0, b]$  such that

$$f = \frac{1}{2}(h - k) \quad \text{and} \quad g = \frac{1}{2}(h + k) .$$

*Proof.*  $1 \iff 2$  The condition (2.1) is equivalent to

$$\begin{aligned} & g(\lambda x + m(1 - \lambda)y) - \lambda^\alpha g(x) - m(1 - \lambda^\alpha)g(y) \\ & \leq \lambda^\alpha f(x) + m(1 - \lambda^\alpha)f(y) - f(\lambda x + m(1 - \lambda)y) \\ & \leq \lambda^\alpha g(x) + m(1 - \lambda^\alpha)g(y) - g(\lambda x + m(1 - \lambda)y) \end{aligned}$$

for all  $x, y \in I$ ,  $\lambda \in [0, 1]$  and  $(\alpha, m) \in [0, 1]^2$ . The two inequalities may be rearranged as

$$(g + f)(\lambda x + m(1 - \lambda)y) \leq \lambda^\alpha (g + f)(x) + m(1 - \lambda^\alpha)(g + f)(y)$$

and

$$(g - f)(\lambda x + m(1 - \lambda)y) \leq \lambda^\alpha (g - f)(x) + m(1 - \lambda^\alpha)(g - f)(y)$$

which are equivalent to the  $(\alpha, m)$ -convexity of  $g + f$  and  $g - f$ , respectively.

$2 \iff 3$  We define the mappings  $f, g$  as  $f = \frac{1}{2}(h - k)$  and  $g = \frac{1}{2}(h + k)$ . Then, if we sum and subtract  $f, g$ , respectively, we have  $g + f = h$  and  $g - f = k$ . By the condition 2 of Lemma 1, the mappings  $g - f$  and  $g + f$  are  $(\alpha, m)$ -convex on  $[0, b]$ , so  $h, k$  are  $(\alpha, m)$ -convex mappings too.  $\square$

**Theorem 5.** Let  $g : [0, \infty) \rightarrow \mathbb{R}$  be an  $(\alpha, m)$ -convex function with  $(\alpha, m) \in (0, 1]^2$ .  $f : [0, \infty) \rightarrow \mathbb{R}$  is  $(g - (\alpha, m))$ -convex dominated mapping and  $0 \leq a < b$ . If  $f \in L_1[a, b] \cap L_1\left[\frac{a}{m}, \frac{b}{m}\right]$ , then the first inequality holds:

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b \frac{f(x) + m(2^\alpha - 1)f\left(\frac{x}{m}\right)}{2^\alpha} dx - f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{1}{b-a} \int_a^b \frac{g(x) + m(2^\alpha - 1)g\left(\frac{x}{m}\right)}{2^\alpha} dx - g\left(\frac{a+b}{2}\right) \end{aligned}$$

and if  $f \in L_1[a, b]$  then the second inequality holds:

$$\begin{aligned} & \left| \frac{1}{2} \left[ \frac{f(a) + mf\left(\frac{a}{m}\right)}{\alpha + 1} + m\alpha \frac{f\left(\frac{b}{m}\right) + mf\left(\frac{b}{m^2}\right)}{\alpha + 1} \right] - \frac{1}{b-a} \int_a^b \frac{f(x) + mf\left(\frac{x}{m}\right)}{2} dx \right| \\ & \leq \frac{1}{2} \left[ \frac{g(a) + mg\left(\frac{a}{m}\right)}{\alpha + 1} + m\alpha \frac{g\left(\frac{b}{m}\right) + mg\left(\frac{b}{m^2}\right)}{\alpha + 1} \right] - \frac{1}{b-a} \int_a^b \frac{g(x) + mg\left(\frac{x}{m}\right)}{2} dx. \end{aligned}$$

*Proof.* By Definition 4 with  $\lambda = \frac{1}{2}$ , as the mapping  $f$  is  $(g - (\alpha, m))$ -convex dominated function, we have that

$$\left| \frac{f(x) + m(2^\alpha - 1)f(y)}{2^\alpha} - f\left(\frac{x + my}{2}\right) \right| \leq \frac{g(x) + m(2^\alpha - 1)g(y)}{2^\alpha} - g\left(\frac{x + my}{2}\right)$$

for all  $x, y \in [0, \infty)$  and  $(\alpha, m) \in (0, 1]^2$ . If we choose  $x = ta + (1-t)b$ ,  $y = (1-t)\frac{a}{m} + t\frac{b}{m}$  and  $t \in [0, 1]$ , then we get

$$\begin{aligned} & \left| \frac{f(ta + (1-t)b) + m(2^\alpha - 1)f\left(\frac{(1-t)a+tb}{m}\right)}{2^\alpha} - f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{g(ta + (1-t)b) + m(2^\alpha - 1)g\left(\frac{(1-t)a+tb}{m}\right)}{2^\alpha} - g\left(\frac{a+b}{2}\right). \end{aligned}$$

Integrating over  $t$  on  $[0, 1]$  we deduce that

$$\begin{aligned} & \left| \frac{\int_0^1 f(ta + (1-t)b) dt + m(2^\alpha - 1) \int_0^1 f\left(\frac{(1-t)a+tb}{m}\right) dt}{2^\alpha} - f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{\int_0^1 g(ta + (1-t)b) dt + m(2^\alpha - 1) \int_0^1 g\left(\frac{(1-t)a+tb}{m}\right) dt}{2^\alpha} - g\left(\frac{a+b}{2}\right) \end{aligned}$$

and so the first inequality is proved.

Since  $f$  is  $(g - (\alpha, m))$ -convex dominated function, we have

$$\begin{aligned} & |t^\alpha f(x) + m(1-t^\alpha)f(y) - f(tx + m(1-t)y)| \\ & \leq t^\alpha g(x) + m(1-t^\alpha)g(y) - g(tx + m(1-t)y), \text{ for all } x, y > 0 \end{aligned}$$

which gives for  $x = a$  and  $y = \frac{b}{m}$

$$\begin{aligned} (2.2) \quad & \left| t^\alpha f(a) + m(1-t^\alpha)f\left(\frac{b}{m}\right) - f\left(ta + m(1-t)\frac{b}{m}\right) \right| \\ & \leq t^\alpha g(a) + m(1-t^\alpha)g\left(\frac{b}{m}\right) - g\left(ta + m(1-t)\frac{b}{m}\right) \end{aligned}$$

and for  $x = \frac{a}{m}$ ,  $y = \frac{b}{m^2}$  and then multiply with  $m$

$$\begin{aligned} (2.3) \quad & \left| mt f\left(\frac{a}{m}\right) + m^2(1-t)f\left(\frac{b}{m^2}\right) - m f\left(t\frac{a}{m} + (1-t)\frac{b}{m}\right) \right| \\ & \leq mt g\left(\frac{a}{m}\right) + m^2(1-t)g\left(\frac{b}{m^2}\right) - mg\left(t\frac{a}{m} + (1-t)\frac{b}{m}\right) \end{aligned}$$

for all  $t \in [0, 1]$ . By properties of modulus, if we add the inequalities in (2.2) and (2.3), we get

$$\begin{aligned} & \left| t^\alpha \left[ f(a) + mf\left(\frac{a}{m}\right) \right] + m(1-t^\alpha) \left[ f\left(\frac{b}{m}\right) + mf\left(\frac{b}{m^2}\right) \right] \right. \\ & \quad \left. - \left[ f\left(ta + m(1-t)\frac{b}{m}\right) + mf\left(t\frac{a}{m} + (1-t)\frac{b}{m}\right) \right] \right| \\ & \leq t^\alpha \left[ g(a) + mg\left(\frac{a}{m}\right) \right] + m(1-t^\alpha) \left[ g\left(\frac{b}{m}\right) + mg\left(\frac{b}{m^2}\right) \right] \\ & \quad - \left[ g\left(ta + m(1-t)\frac{b}{m}\right) + mg\left(t\frac{a}{m} + (1-t)\frac{b}{m}\right) \right]. \end{aligned}$$

Thus, integrating over  $t$  on  $[0, 1]$  we obtain the second inequality. The proof is completed.  $\square$

**Remark 1.** If we choose  $\alpha = 1$  in Theorem 5, we get two inequalities of Hermite-Hadamard type for functions that are  $(g, m)$ -convex dominated in Theorem 1.

**Theorem 6.** Let  $g : [0, \infty) \rightarrow \mathbb{R}$  be an  $(\alpha, m)$ -convex function with  $(\alpha, m) \in (0, 1]^2$ .  $f : [0, \infty) \rightarrow \mathbb{R}$  is  $(g - (\alpha, m))$ -convex dominated mapping and  $0 \leq a < b$ . If  $f \in L_1[a, b]$ , then the following inequality holds:

$$\begin{aligned} (2.4) \quad & \left| \frac{1}{2} \left[ \frac{f(a) + f(b) + m\alpha f\left(\frac{a}{m}\right) + m\alpha f\left(\frac{b}{m}\right)}{\alpha + 1} \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{1}{2} \left[ \frac{g(a) + g(b) + m\alpha g\left(\frac{a}{m}\right) + m\alpha g\left(\frac{b}{m}\right)}{\alpha + 1} \right] - \frac{1}{b-a} \int_a^b g(x) dx \end{aligned}$$

*Proof.* Since  $f$  is  $(g - (\alpha, m))$ -convex dominated function, we have

$$\begin{aligned} & \left| t^\alpha f(a) + m(1-t^\alpha) f\left(\frac{b}{m}\right) - f\left(ta + m(1-t)\frac{b}{m}\right) \right| \\ & \leq t^\alpha g(a) + m(1-t^\alpha) g\left(\frac{b}{m}\right) - g\left(ta + m(1-t)\frac{b}{m}\right) \end{aligned}$$

and

$$\begin{aligned} & \left| t^\alpha f(b) + m(1-t^\alpha) f\left(\frac{a}{m}\right) - f\left(tb + m(1-t)\frac{a}{m}\right) \right| \\ & \leq t^\alpha g(b) + m(1-t^\alpha) g\left(\frac{a}{m}\right) - g\left(tb + m(1-t)\frac{a}{m}\right) \end{aligned}$$

for all  $t \in [0, 1]$ . Adding the above inequalities, we get

$$\begin{aligned} & \left| t^\alpha [f(a) + f(b)] + m(1-t^\alpha) \left[ f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right) \right] - f\left(ta + m(1-t)\frac{b}{m}\right) - f\left(tb + m(1-t)\frac{a}{m}\right) \right| \\ & \leq t^\alpha [g(a) + g(b)] + m(1-t^\alpha) \left[ g\left(\frac{a}{m}\right) + g\left(\frac{b}{m}\right) \right] - g\left(ta + m(1-t)\frac{b}{m}\right) - g\left(tb + m(1-t)\frac{a}{m}\right). \end{aligned}$$

Integrating over  $t \in [0, 1]$  and then by dividing the resulting inequality with 2, we get the desired result. The proof is completed.

Another proof can be done as the following.

Since  $f$  is  $(g - (\alpha, m))$ -convex dominated, we have by Lemma 1 that  $g + f$  and  $g - f$  are  $(\alpha, m)$ -convex on  $[0, b]$ , and so by the Hadamard's type inequality for  $(\alpha, m)$ -convex functions in (1.4)

$$(2.5) \quad \begin{aligned} & \frac{1}{b-a} \int_a^b (g+f)(x) dx \\ & \leq \frac{1}{2} \left[ \frac{(g+f)(a) + (g+f)(b) + m\alpha(g+f)\left(\frac{a}{m}\right) + m\alpha(g+f)\left(\frac{b}{m}\right)}{\alpha+1} \right] \end{aligned}$$

and

$$(2.6) \quad \begin{aligned} & \frac{1}{b-a} \int_a^b (g-f)(x) dx \\ & \leq \frac{1}{2} \left[ \frac{(g-f)(a) + (g-f)(b) + m\alpha(g-f)\left(\frac{a}{m}\right) + m\alpha(g-f)\left(\frac{b}{m}\right)}{\alpha+1} \right] \end{aligned}$$

By using the inequalities in (2.5) and (2.6), we get the inequality in (2.4).  $\square$

### 3. $(g, r)$ -CONVEX DOMINATED FUNCTIONS

**Definition 5.** Let positive function  $g : [a, b] \rightarrow \mathbb{R}$  be a given  $r$ -convex function on  $[a, b]$ . The real function  $f : [a, b] \rightarrow \mathbb{R}$  is called  $(g, r)$ -convex dominated on  $[a, b]$  if the following condition is satisfied:

$$\begin{aligned} & |M_r(f(x), f(y); \lambda) - f(\lambda x + (1-\lambda)y)| \\ & \leq M_r(g(x), g(y); \lambda) - g(\lambda x + (1-\lambda)y) \end{aligned}$$

for all  $x, y \in [a, b]$  and  $\lambda \in [0, 1]$ .

**Theorem 7.** Let positive function  $g : [a, b] \rightarrow \mathbb{R}$  be an  $r$ -convex function on  $[a, b]$ .  $f : [a, b] \rightarrow \mathbb{R}$  is  $(g, r)$ -convex dominated mapping and  $0 \leq a < b$ . If  $f \in L_1[a, b]$ , then the following inequality holds:

$$\left| L_r(f(a), f(b)) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq L_r(g(a), g(b)) - \frac{1}{b-a} \int_a^b g(x) dx$$

for all  $x, y \in I$ ,  $\lambda \in [0, 1]$  and  $L_r(f(a), f(b))$  as in (1.6).

*Proof.* By the Definition 5 with  $r = 0$ ,  $f(a) \neq f(b)$ , we have

$$\begin{aligned} & |f^\lambda(a) f^{1-\lambda}(b) - f(\lambda a + (1-\lambda)b)| \\ & \leq g^\lambda(a) g^{1-\lambda}(b) - g(\lambda a + (1-\lambda)b). \end{aligned}$$

Integrating the above inequality over  $\lambda$  on  $[0, 1]$ , we have

$$\begin{aligned} & \left| f(b) \int_0^1 \left[ \frac{f(a)}{f(b)} \right]^\lambda d\lambda - \int_0^1 f(\lambda a + (1-\lambda)b) d\lambda \right| \\ & \leq g(b) \int_0^1 \left[ \frac{g(a)}{g(b)} \right]^\lambda d\lambda - \int_0^1 g(\lambda a + (1-\lambda)b) d\lambda. \end{aligned}$$

By a simple calculation we have

$$\begin{aligned} & \left| \frac{f(b) - f(a)}{\ln f(b) - \ln f(a)} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{g(b) - g(a)}{\ln g(b) - \ln g(a)} - \frac{1}{b-a} \int_a^b g(x) dx. \end{aligned}$$

The above inequality can be written as

$$\left| L_r(f(a), f(b)) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq L_r(g(a), g(b)) - \frac{1}{b-a} \int_a^b g(x) dx.$$

For  $r = 0$ ,  $f(a) = f(b)$ , we have with the same development

$$\begin{aligned} & |f(a) - f(\lambda a + (1-\lambda)b)| \\ & \leq g(a) - g(\lambda a + (1-\lambda)b) \end{aligned}$$

and this inequality can be written as

$$\left| L_r(f(a), f(b)) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq L_r(g(a), g(b)) - \frac{1}{b-a} \int_a^b g(x) dx.$$

By the Definition 5 with  $r \neq 0, -1$ ,  $f(a) \neq f(b)$ , we have

$$\begin{aligned} & \left| (\lambda f^r(a) + (1-\lambda)f^r(b))^{\frac{1}{r}} - f(\lambda a + (1-\lambda)b) \right| \\ & \leq (\lambda g^r(a) + (1-\lambda)g^r(b))^{\frac{1}{r}} - g(\lambda a + (1-\lambda)b). \end{aligned}$$

Integrating the above inequality over  $\lambda$  on  $[0, 1]$ , we have

$$\begin{aligned} & \left| \frac{r}{r+1} \frac{f^{r+1}(a) - f^{r+1}(b)}{f^r(a) - f^r(b)} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{r}{r+1} \frac{g^{r+1}(a) - g^{r+1}(b)}{g^r(a) - g^r(b)} - \frac{1}{b-a} \int_a^b g(x) dx. \end{aligned}$$

The above inequality can be written as

$$\begin{aligned} & \left| L_r(f(a), f(b)) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq L_r(g(a), g(b)) - \frac{1}{b-a} \int_a^b g(x) dx. \end{aligned}$$

For  $r \neq 0$  and  $f(a) = f(b)$ , we have similarly

$$\begin{aligned} & \left| (f^r(a))^{\frac{1}{r}} - f(\lambda a + (1-\lambda)b) \right| \\ & \leq (g^r(a))^{\frac{1}{r}} - g(\lambda a + (1-\lambda)b). \end{aligned}$$



Then integrating the above inequality over  $\lambda$  on  $[0, 1]$ , we have

$$\begin{aligned} & \left| L_r(f(a), f(b)) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq L_r(g(a), g(b)) - \frac{1}{b-a} \int_a^b g(x) dx. \end{aligned}$$

Finally, let  $r = -1$ . For  $f(a) \neq f(b)$  we have again

$$\begin{aligned} & \left| (\lambda f^{-1}(a) + (1-\lambda)f^{-1}(b))^{-1} - f(\lambda a + (1-\lambda)b) \right| \\ & \leq (\lambda g^{-1}(a) + (1-\lambda)g^{-1}(b))^{-1} - g(\lambda a + (1-\lambda)b). \end{aligned}$$

Integrating the above inequality over  $\lambda$  on  $[0, 1]$ , we have

$$\begin{aligned} & \left| \frac{f(a)f(b)}{f(b)-f(a)} \int_{\frac{1}{f(a)}}^{\frac{1}{f(b)}} \lambda^{-1} d\lambda - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{g(a)g(b)}{g(b)-g(a)} \int_{\frac{1}{f(a)}}^{\frac{1}{f(b)}} \lambda^{-1} d\lambda - \frac{1}{b-a} \int_a^b g(x) dx. \end{aligned}$$

The above inequality can be written as

$$\begin{aligned} & \left| L_{-1}(f(a), f(b)) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq L_{-1}(g(a), g(b)) - \frac{1}{b-a} \int_a^b g(x) dx. \end{aligned}$$

The proof is completed.  $\square$

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